Parabolic vector bundles

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A Tribute to C.S. Seshadri
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Papers of C.S. Seshadri on parabolic bundles

1. Moduli of π-vector bundles over an algebraic curve. 1970 *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, 139–260.
2. Moduli of vector bundles on curves with parabolic structures
   - *Bull. Amer. Math. Soc.* **83** (1977), 124–126.
   - (with V. Mehta) *Math. Ann.* **248** (1980), 205–239.
3. (with V. Balaji) Parahoric $\mathcal{G}$-torsors on a compact Riemann surface *J. Algebraic Geom.* **24** (2015), 1–49.

Definitions

Let π be a group acting on a manifold X.

A **π-vector bundle** is a vector bundle E on X together with an action of π on E, such that the projection $E \longrightarrow X$ is equivariant.

Let X be a smooth curve, $D = \{x_1, \dots, x_n\} \subset X$ distinct points.

A **parabolic vector bundle** over (X, D) is a vector bundle E over X together with a weighted flag over the fiber $E|_x$ for each $x \in D$ called parabolic structure, i.e., a filtration $F_\bullet E_x$ by linear subspaces

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x+1} = 0$$

together with a system of real weights $0 \leq \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,l_x} < 1$.

These notions stem from generalizations of Narasimhan-Seshadri's theorem.

Background

Consider the universal cover of a compact Riemann surface

$$\widetilde{X} \longrightarrow X = \widetilde{X}/\pi_1(X)$$

Given an (irreducible) unitary representation of $\pi_1(X)$ in $\mathrm{U}(r)$, the trivial bundle $\widetilde{X} \times \mathbb{C}^r$ has a natural structure of $\pi_1(X)$-bundle, and the quotient is a vector bundle E on X. A vector bundle constructed in this way is called an **(irreducible) unitary bundle**.

For line bundles, unitary line bundles are exactly line bundle of degree 0.
What about higher rank?

A vector bundle E on X is **stable** (resp. semistable) if for all proper subbundles $E' \subset E$,

$$\frac{\deg E'}{\operatorname{rk} E'} < \frac{\deg E}{\operatorname{rk} E} \qquad (\text{resp. } \leq)$$

Proposition-Definition (Jordan-Hölder filtration)

E semistable bundle. There is a filtration by semistable bundles

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq E_3 \subsetneq \cdots \subsetneq E_l = E$$

such that all quotients E_i/E_{i-1} are stable. The filtration is not necessarily unique, but the graded object

$$\mathrm{gr}^{JH}(E) = \bigoplus_{i=1}^{l} E_i/E_{i-1}$$

is unique up to isomorphism.

Definition (S-equivalence)

Two semistable bundles are S-equivalent if and only if the graded objects are isomorphic

$$\mathrm{gr}^{JH}(E) \cong \mathrm{gr}^{JH}(E')$$

Remark: If E is stable, then the filtration has only one element, and $\mathrm{gr}^{JH}(E) = E$

Theorems of Narasimhan-Seshadri

Theorem I

A vector bundle is stable of degree 0 if and only if it is irreducible unitary.

To allow arbitrary degree, use extension of $\pi_1(X)$ (geometrically, cover of X ramified over a single point).

Theorem II

There exists a projective moduli space of unitary representations of the fundamental group of X.

To prove Theorem II one actually constructs, using GIT, a moduli space of S-equivalence classes of semistable vector bundles.

$\{$unitary representations of $\pi\} \Leftrightarrow \{$semistable bundles of deg $0\}/$S-equiv.

- $\mathbb{H} = \{x + yi \in \mathbb{C} : y > 0\}$, the upper half plane
- $\Gamma \subset PSL(2, \mathbb{R})$ a Fuchsian group acting properly discontinuously on $\mathbb{H}$

1. Suppose the action is free and $\mathbb{H}/\Gamma = X$ is a compact Riemann surface. Then $\mathbb{H}$ is the universal cover, $\Gamma = \pi_1(X)$, and we are in the previous situation.
2. In [Seshadri-70], a nonfree action is allowed, but we still assume $\mathbb{H}/\Gamma$ is compact.
3. In [Seshadri-77, Mehta-Seshadri-80] we allow $\mathbb{H}/\Gamma$ not compact, but assume finite volume.

In other words, **parabolic vector bundles** appear in a natural way when we consider **non-free actions** of Γ in $\mathbb{H}$.

If $\mathbb{H}/\Gamma$ is **compact**, then Γ has a presentation with generators

$$A_i,\ B_i\ (i=1,\dots,g), \quad C_j\ (j=1,\dots s)$$

and relations (with integers $m_j \geq 2$)

$$\prod_i [A_i, B_i] \prod_j C_j = 1, \quad C_j^{m_j} = 1 \quad (j=1,\dots,s)$$

(This will produce rational parabolic weights)

If $\mathbb{H}/\Gamma$ has **finite volume**, then Γ has a presentation with generators

$$A_i,\ B_i\ (i=1,\dots,g), \quad C_j\ (j=1,\dots s), \quad \mathbf{D_k}\ (\mathbf{k}=\mathbf{1},\dots \mathbf{t})$$

and relations (with integers $m_j \geq 2$)

$$\prod_i [A_i, B_i] \prod_j C_j \prod_{\mathbf{k}} \mathbf{D_k} = 1, \quad C_j^{m_j} = 1 \quad (j=1,\dots,s)$$

(This will allow real parabolic weights)

Suppose $Y = \mathbb{H}/\Gamma$ is **compact**. Then there exists a subgroup $\Gamma_0 \subset \Gamma$ of finite index acting freely

$$\begin{array}{ccc} & & \mathbb{H} \\ & {\scriptstyle \Gamma_0} \swarrow & \downarrow {\scriptstyle \Gamma} \\ X & \xrightarrow[\pi=\Gamma/\Gamma_0]{} & Y \end{array}$$

The study of Γ-vector bundles on $\mathbb{H}$ reduces to the study of π-bundles on the projective curve X, so it is an algebraic problem.

An irreducible π-unitary vector bundle on X is a vector bundle coming from an irreducible unitary representation of Γ.

The definition of π-(semi)stability is the usual one, but we only check with π-invariant vector subbundles.

Theorem I

A π-vector bundle is stable of degree 0 if and only if it is irreducible π-unitary.

An irreducible π-unitary vector bundle on X is a vector bundle coming from an irreducible unitary representation of Γ.

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Theorem I

A π-vector bundle is stable of degree 0 if and only if it is irreducible π-unitary.

We say that two π-bundles E_1 and E_2 on X have the same type if for all $x \in X$ there is an open neighborhood U of x invariant under π_x, the isotropy subgroup of x, such that their restrictions to U are isomorphic as π_x-bundles.

Theorem II

There exists a projective moduli space of π-unitary vector bundles on X of fixed type τ.

Idea of proof of Theorem I

1. π-unitary bundle is π-semistable of degree 0
2. irreducible π-bundle is π-stable of degree 0
3. $\{\pi\text{-unitary}\} \subset \{\pi\text{-stable of deg } 0\}$ is a **closed** subset
    1. semicontinuity of coherent cohomology
    2. $\mathrm{U}(r)$ is compact
    3. nonzero map between a semistable and a stable bundle of deg 0 is isomorphism
4. $\{\text{irreducible } \pi\text{-unitary}\} \subset \{\text{vector bundles}\}$ is an **open** subset
    - Let $\rho : \pi \longrightarrow \mathrm{U}(r)$ irreducible, E the associated vector bundle
    - There exists a smooth local moduli space V of equivalence classes of irreducible π-unitary representations around ρ of dimension $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} H^1(X, \pi, E^\vee \otimes E)$. (universal local property).
    - E is stable, hence simple. There exists a local moduli D of simple bundles around E of dimension $\dim_{\mathbb{C}} D = \dim_{\mathbb{C}} H^1(X, \pi, E^\vee \otimes E)$
5. Since the unitary condition is both open and close, it is enough to produce a connected family which parametrizes all π-stable bundles (of fixed type τ). This is constructed using Quot-schemes.

Idea of proof of Theorem II

- Let $\mathcal{V} = \mathcal{O}_X(-m)^{\oplus p}$, ($m \gg 0$ fixed to be determined in the proof)
- $\big(E, \varphi : \mathbb{C}^p \xrightarrow{\cong} H^0(E(m))\big)$, E π-vector bundle, gives a quotient

$$q : \mathcal{V} \overset{\varphi}{\cong} H^0(E(m)) \otimes \mathcal{O}_X(-m) \twoheadrightarrow E$$

  hence a point in $Q := \mathrm{Quot}(\mathcal{V}, p)$.
- The group π acts on Q, and $q \in Q^\pi$, the π-invariant subscheme.
- Fix N points $T = \{t_1, \ldots, t_N\} \subset X$. For each $t \in T$, q induces a quotient on fibers

$$q_t : \mathbb{C}^p \twoheadrightarrow E_t$$

- Define a polarization on Q^π using $Q^\pi \longrightarrow \prod_{t \in T} \mathrm{Gr}(\mathbb{C}^p, r)$ and the polarization $(1, \ldots, 1)$ on the product.
- π-(semi)stability $\Leftrightarrow$ GIT-(semi)stability
- The moduli space is the GIT quotient by $\mathrm{SL}(p)$ using the induced polarization.

Parabolic bundles [Seshadri-77, Mehta-Seshadri-80]

Suppose $U = \mathbb{H}/\Gamma$ has **finite volume**. U is a (dense) open subset on a smooth projective curve X

$$\begin{array}{ccc} \mathbb{H} & \hookrightarrow & \mathbb{H}^+ \\ \Gamma \downarrow & & \downarrow \Gamma \\ U & \hookrightarrow & X \end{array} \qquad \mathbb{H}^+ = \mathbb{H} \cup \{\text{parabolic cusps}\}$$

Parabolic points of $D = \{x_1, \ldots, x_n\} \subset X$ are the images of fixed points of Γ in $\mathbb{H}^+$ (i.e., both parabolic and elliptic fixed points of Γ)

A **parabolic vector bundle** over (X, D) is a vector bundle E over X together with a weighted flag over the fiber $E|_x$ for each $x \in D$ called parabolic structure, i.e., a filtration $F_\bullet E_x$ by linear subspaces

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x+1} = 0$$

together with a system of real weights $0 \le \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,l_x} < 1$.

Parabolic bundles [Seshadri-77, Mehta-Seshadri-80]

Parabolic structure on elliptic points

- Let $\Gamma \longrightarrow GL(r, \mathbb{C})$ be a representation. Consider the Γ-bundle $\widetilde{E} = \mathbb{H} \times \mathbb{C}^r$.
- Let $y \in \mathbb{H}$ be a point where the action is not free, and $x \in \mathbb{H}/\Gamma$ the image.
- Let $E = p_*^\Gamma \widetilde{E}$ be the sheaf of invariant sections.
- The isotropy π_y is finite cyclic of some order N. It acts on E_y as

$$\begin{pmatrix} e^{2\pi i d_1/N} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i d_r/N} \end{pmatrix}$$

  which gives a filtration by eigenspaces

$$\widetilde{E}|_x = \widetilde{E}_{x,1} \supsetneq \widetilde{E}_{x,2} \supsetneq \cdots \supsetneq \widetilde{E}_{x,l_x+1} = 0$$

  with rational weights $0 \leq \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,l_x} < 1$, given by the different values of d_j/N.
- It can be seen that this filtration induces a filtration on E_x

Parabolic bundles [Seshadri-77, Mehta-Seshadri-80]

Parabolic structure on parabolic points

- Let $\Gamma \longrightarrow \mathrm{U}(r)$ be a unitary representation, $\mathbb{H}/\Gamma$ not compact.
- $\mathbb{H}/\Gamma = U \subset X = \mathbb{H}^+/\Gamma$. Let $x \in X \setminus U$ and $y \in \mathbb{H}^+$. We may assume $y = \infty$.
- The vector bundle $p_*^\Gamma \widetilde{E}$ on U can be extended canonically to a vector bundle E on X:
  - Let $\mathbb{H}_\delta = \{z : \mathrm{Im}(z) > \delta \geq 0\}$.
  - $V_\delta = \mathbb{H}_\delta/\mathbb{Z} \cup \{x\}$ is an open neighborhood of x ($\mathbb{Z}$ acts as $z \mapsto z+1$)
  - A section of $E|_{V_\delta}$ is a **bounded $\mathbb{Z}$-invariant section** of $\mathbb{H}_\delta \times \mathbb{C}^r$ (Γ acts on $\mathbb{C}^r$ by the given action)
  - The germ of E at x is generated (as $\mathcal{O}_{X,x}$-module) by sections of the form $e^{2\pi i \alpha_j z} e_j$, $0 \leq \alpha_1 < \cdots < \alpha_r < 0$.
- As in the previous case, it can be shown that we can define a filtration on $E|_x$ with weights α_i.

Parabolic bundles [Seshadri-77, Mehta-Seshadri-80]

Let E be a parabolic vector bundle. A vector subbundle $E' \subset E$ inherits a canonical structure of parabolic bundle

$$F_\bullet E'_x = E'_x \cap F_\bullet E_x \quad \alpha'_{x,i} = \max\{\alpha_{x,j} : E'_x \cap E_{x,j} = E'_{x,i}\}$$

$$\text{pardeg}\, E = \deg E + \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_i (\dim E_{x,i} - \dim E_{x,i+1})$$

Definition

A vector bundle E on a smooth projective curve X is stable (resp. semistable) if for all proper subbundles $E' \subset E$,

$$\frac{\text{pardeg}\, E'}{\text{rk}\, E'} < \frac{\text{pardeg}\, E}{\text{rk}\, E} \qquad (\text{resp. } \leq)$$

S-equivalence is defined as in the case of vector bundles.

Theorem I-a

A parabolic vector bundle is stable of degree 0 if and only if it comes from an irreducible unitary representation of Γ.

S-equivalence classes of semistable parabolic bundles of degree 0 correspond to unitary representations.

Theorem I-b

Let X be a compact Riemann surface and $D = \{x_1, \ldots, x_n\}$ distinct points. There is a bijection between unitary representations of $\pi_1(X - D)$ such that the holonomy around x_i is conjugate to a diagonal matrix

$$\begin{pmatrix} e^{2\pi i \alpha_{x,1}} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i \alpha_{x,r}} \end{pmatrix} \quad 0 \leq \alpha_{x,1} < \cdots < \alpha_{x,r} < 1$$

and S-equivalence classes of semistable parabolic bundles with weights $\alpha_{x,i}$.

Irreducible representations correspond to stable parabolic bundles.

Theorem II

Let X be a compact Riemann surface, and $D = \{x_1, \dots, x_n\}$ distinct points. Fix the type of the parabolic structure (i.e., the dimensions of the quotients of the filtrations on the fibers of each $x_i \in D$). Fix weights α_{x_i}. Fix the rank and the parabolic degree.

There is a projective moduli space of S-equivalence classes of semistable parabolic bundles.

Remarks

- The notion of (semi)stability depends on the weights $\alpha_{x,i}$, which can be thought of as parameters for the notion of (semi)stability.
- For any choice of **real** weights $\alpha_{x,i}$, there is a set of **rational** weights $\alpha'_{x,i}$ which gives the same notion of (semi)stability. Therefore, we may assume the weights are rational.

- Let $\mathcal{V} = \mathcal{O}_X(-m)^{\oplus p}$, ($m \gg 0$ fixed to be determined in the proof)

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- $\big(E, \varphi : \mathbb{C}^p \xrightarrow{\cong} H^0(E(m))\big)$, E parabolic bundle, gives a quotient

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- For each $t \in T$, q induces a quotient $\mathbb{C}^p \twoheadrightarrow E|_t$
- For each $x \in D$, and j, the parabolic filtration induces a quotient $E|_x \twoheadrightarrow E/E_{x,j}$ (of dimension $d_{x,j}$).

$$\prod_{t \in T} \mathrm{Gr}(\mathbb{C}^p, r) \times \prod_{x \in D} \prod_j \mathrm{Gr}(E_x, d_{x,j})$$

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More precisely, we get a point in

$$\prod_{x \in D} \prod_j \mathrm{Gr}(\mathcal{E}_x, d_{x,j}) \quad \mathcal{E}_x = \mathcal{E}|_{\mathrm{Quot}(\mathcal{V},p) \times \{x\}},\ \mathcal{E} \text{ universal quotient for } \mathrm{Quot}(\mathcal{V}, p)$$

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- Define a polarization using the weights $\alpha_{x,j}$ (which we assume rational). The moduli space is the GIT quotient by the action of $\mathrm{SL}(p)$.

Alternative definition of parabolic bundles

Definition as sheaf filtration (Simpson, Maruyama-Yokogawa)

We can define the parabolic bundle structure at each $x \in D$ as a sheaf filtration by vector bundles

$$E = E_1^x \supsetneq E_2^x \supsetneq E_3^x \supsetneq \cdots \supsetneq E_l^x \supsetneq E(-x)$$

together with weights $0 \leq \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,l_x} < 1$

Relationship between both definitions

$$0 \longrightarrow E_i^x \longrightarrow E \longrightarrow E_x/E_{x,i} \longrightarrow 0$$

Recall:

- E_i^x are vector bundles on X.
- $E_{x,i}$ are vector subspaces of the fiber E_x of E at x.

Equivariant bundles and parabolic bundles

- Let X be a variety with an action of a finite group π, and consider the quotient
$$p : X \longrightarrow Y = X/\pi$$
If the isotropy groups are cyclic, then π-vector bundle E on X gives a parabolic vector bundle structure on $F = p_*^\pi(E)$ (π-invariant direct image) on Y (with rational weights).
- Conversely, given a parabolic bundle on a variety Z (rational weights), we can construct a π-vector bundle on a π-variety $\widetilde{Z}$ such that the parabolic bundle comes as above.
- This correspondence is implicit in the work of Seshadri. As far as I know, the first explicit description is in a paper of Usha Bhosle (Math. Annalen, 1984).

  A π-vector bundle E gives a representation of the isotropy group π_x on the fiber E_x. The eigenspace decomposition gives a filtration on E_x and, after some work, we obtain a filtration on F_x.

Equivariant bundles and parabolic bundles

- Indranil Biswas (Duke, 1997) gives a formula for this correspondence in terms of sheaf filtrations by vector bundles

$$F = p_*^{\pi}(E), \quad F_i^x = p_*^{\pi}(E(-ix)), \quad i = 0, \dots, |\pi_x|,$$

  generalizes this correspondence for higher dimensional varieties, and in a series of papers finds many applications.
- Mundet i Riera (IJM, 2002) gives another construction, using a fibration instead of a covering, which does not depend on the weights.

Properties of the moduli space of parabolic bundles

There is a lot of work studying moduli spaces of parabolic bundles. Just to mention some of it:

- Rationality of the moduli space (Boden-Yokogawa, 1999)
- Betti numbers (Nitsure 1996, Holla 2000)
- Betti numbers of parabolic Higgs bundles (Garcia–Prada-Gothen-Muñoz, 2007)
- Torelli theorem (rank 2 Balaji-del Baño-Biswas 2001, rank n Alfaya, G. 2019) and automorphisms.
- Symplectic and Poisson geometry of the moduli space of parabolic Higgs bundles (Hurtubise 1996, Bottacin 2000, Logares-Martens 2010)

The notion of parabolic vector bundle has been generalized in many directions. I will just mention a few.

- Parabolic bundles can be defined for X variety with $\dim X > 1$. A parabolic structure is given by a divisor D (usually a union of smooth divisors with normal crossings), and a vector bundle filtration of the vector bundle $E|_D$ with real weights [Maruyama-Yokogawa 1992, Bhosle 1992]
- A related notion (**Quasi-parabolic bundles**) is used to study vector bundles on nodal curves (Bhosle, Nagaraj-Seshadri, ...).
- Parabolic bundle structures have been defined for **principal bundles** (Teleman-Woodward, Balaji, Biswas, Nagaraj). Another definition is used in the geometric Langlands program of Beilinson-Drinfeld. Also Balaji and Seshadri (JAG, 2015) have used parahoric $\mathcal{G}$-bundles as a generalization in this direction.

- Let X be a Riemann surface. If X is compact and we consider representations of $\pi_1(X)$ into the **linear group** $\mathrm{GL}(n)$ we are lead to Higgs bundles.

  If $U = X \setminus \{x_1, \ldots, x_n\}$, representations of $\pi_1(U)$ in $\mathrm{GL}(n)$ leads us to parabolic Higgs bundles. Simpson (J. AMS, 1990) makes a very detailed study in this situation, using tame harmonic bundles on U to relate:

  1. Parabolic Higgs bundles on X
  2. Filtered regular $\mathcal{D}$-modules on U
  3. Filtered local systems on U

- **Parabolic Higgs bundles for higher dimensional varieties** have been studied by Biquard, Jost-Zuo, and more recently by Takuro Mochizuki (first for tame harmonic bundles and then for wild harmonic bundles).
- The work of Mochizuki is applied by Donagi and Pantev in their study of the **geometric Langlands program**.

Thank you