C.S. Seshadri, my mentor, mathematician and a remarkable person

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Seshadri was my mentor and teacher for well over 3 decades; I joined him as a doctoral student towards the end of 1984. As a teacher he was very distinctive, somewhat indulgent to start with, but pressure and demand to keep pace with him increased with rapidity.

He set very high standards for himself and hence on anyone whom he considered proximate.

There was a constant pressure, both on himself and his collaborators to seek further and better any perspective gained.

There is a quote from one of the oldest texts, the Rig Veda, which very aptly describes Seshadri's drive:

"As one climbs from peak to peak, there becomes clear to the view the much that still needs to be done".

I am indebted to him for being my master in mathematics. He had very strange modes of communication, of conveying what he felt were the core issues, but he always managed to get it across and I became richer after every such transaction. Half sentences, empty pauses and bursts of non-trivial technical stuff was the typical process.

I had the special privilege of being able to read off his "un-expressed" ideas (*anāhata nāda*, or the *"unstruck note"*) and then communicate to younger colleagues.

A humorous incident: once I was with two younger colleagues in Seshadri's office and Seshadri was explaining very keenly some aspects of Schubert geometry. It was in the morning, the black-board had been cleaned the previous day and there was no chalk nearby to be found. So, Seshadri quickly went to the board, and not finding the chalk, undeterred, he started writing with his fingers, detailed points 1) , 2), 3) and 4). The two younger students were gazing at the empty board, at the detailed ideas stringed out in invisible letters. Seshadri was also explaining simultaneously the main ideas in half-sentences and quiet pauses!

Seshadri was also gazing at the board. Suddenly, he went to the board and erased the line 3) (which was already invisible since it was'nt written with a physical chalk!). Then he said, "Yes!, these three are the main issues, you can go back and check them!" The two colleagues looked at each other and turned to look at me with dismay! I quietly asked both of them to come to my room and could convey to them in my own way the invisible statements and the unsaid ideas! I believe it helped!

He had the uncanny instinct for the key idea and he had perfected a unique non-canonical approach to get to it.

Let me again bring some humour here (knowing him, I am sure he would have enjoyed this part!). Several years back, I was with Seshadri, Lakshmibai, Sundari (Seshadri's wife), Tadao Oda and Christopher Soule and a few others. There was some humorous comment about Seshadri. So I said I wished to tell a joke (which I am sure well known in its original form!). This loosely went as follows: there were two mathematicians, M(1) and M(2) who were friends for years and on an emotional moment, they even told each other that even death will not be able to stop their regular meetings and discussions. As things happen, tragically M(1) passed away. Several days passed and there was no sign of the promised communication! then, one fine day communication got established and M(2) was delighted to speak with disembodied M(1). M(1) recounted many happenings up in heaven, talks by mathematical giants of yester years, and he exclaimed, "I even saw Gauss and Riemann the other day!" and then paused and said in a sombre tone "I have a bit of bad news for you though!" M(2) got worried and immediately responded, "what is that!". M(1) said, "I just now passed by the Notice Board and saw the announcement of your talk here next week!"

All the people, Oda...laughed and Seshadri laughed even more! I had the strange feeling that something non-canonical is brewing here! So I paused and then asked Seshadri, why was that a bad news for M(2), and here was Seshadri's unique response!

Seshadri: "Obviously a bad news! just imagine giving a seminar talk with Gauss and Riemann in the audience!"

The gathering was stunned at this unique interpretation and yet another exhibition of Seshadri's non-canonical thinking went into the history books!

At the same time he would be thrilled when he saw a new insight which he had missed and showed his appreciation very openly. There was a complete awareness of his own stature while being modest and humble at the same time.

My discussions with him in person and over the phone (which often went on till very late in the evening) were a mix of light and groping in the dark, but invariably led to new heights in understanding.

His passion and appetite for math were insatiable. In the past couple of years he was keen to return to p-adic uniformization, Mumford curves and an old paper of Faltings.

My last discussion with him was on 15 July when I lectured to him for an hour. His keenness was intact and it was "discussion as usual" , his failing health notwithstanding. He expressed joy on a new perspective on one of his old passions (bundles on nodal curves) and said " Ah! I missed this, so simple no?"

I said we will continue the next day and he most uncharacteristically responded with a laugh and said " can't give a guarantee for that!"

Before I speak on some of Seshadri's classic papers on "quotients and GIT", I will recount an incident (which was about 5 years ago) to illustrate his insight. Seshadri was very keen to see the solution to the problem of "degeneration of the moduli space of principal G-bundles". In fact, my paper with him "parahoric bundles" was partly motivated by this question.

I now take up two papers of Seshadri, the first entitled "Some results on the quotient space by an algebraic group of automorphisms" and the second being "Quotient spaces modulo reductive algebraic groups" to which I will return later. The aspect that I wish to highlight here is somewhat general and does not really require the group to be reductive or even affine.

Question: *Let X be a scheme on which a connected algebraic group acts properly. Then does the geometric quotient X/G exist?*

Recall that a G-morphism $f : X \to Y$ is called a *good quotient* if (1) f is a surjective affine G-invariant morphism, (2) $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$ and (3) f sends closed G-stable subsets to closed subsets and separates disjoint closed G-stable subsets of X. The quotient f is called a *geometric quotient* if it is a good quotient and moreover for each $x \in X$, the G-orbit $G.x$ is closed in X.

It is known that the question as stated above fails in general but Seshadri gave some very general conditions under which it holds. He proves the following theorem: let X be a normal scheme of finite type (or more generally a normal algebraic space of finite type over k) and G a connected affine algebraic group acting properly on X. The the geometric quotient X/G exists as a normal algebraic space of finite type. When the action is proper, a geometric quotient is simply a topological quotient with the property (2) above.

The technique of *elimination of finite isotropies*: let X be an irreducible excellent scheme over k and G affine algebraic group acting properly on X. Then there is a diagram:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ Z & & \end{array}$$

where Y is irreducible and G acts properly on Y. Further, p is a Zariski locally trivial principal G-bundle and q a finite dominant G-morphism with Y/X Galois with Galois group Γ whose action on Y commutes with the G-action.

Seshadri's results can be easily seen to work in the relative case, i.e. for normal algebraic spaces X over excellent schemes S and G_S an affine group scheme over S.

These ideas are central to the major developments by Kollar and Keel and Mori on "Quotients" in the nineties.

I come to a few of Seshadri's papers in the subject of Geometric Invariant Theory. Apart from the first one discussed above, where Seshadri introduced S-equivalence.

The first one was Mumford's conjecture for $GL(2)$ which apart from proving the conjecture gave a restricted "valuative criterion" which predates the famous Langton criterion.

He needed to to show, what he called the "covariant" from R^{ss} (the open subset of semistable bundles in the Quot scheme) to a product of Grassmannians is an imbedding and the central issue was the properness of the covariant. Seshadri introduces a new amazing device namely, a multiple-valued mapping to prove the properness.

This approach of Seshadri's became the standard prototype for all moduli constructions, the most general one being the one by Simpson in the early nineties.

Geometric reductivity:

Let G *be a reductive algebraic group over an algebraically closed field* k. *Then* G *is* geometrically reductive *if, for every finite-dimensional rational* G*-module* V *and a* G*-invariant point* $v \in V$, $v \neq 0$, *there is a* G*-invariant homogeneous polynomial* F *on* V *of positive degree such that* $F(v) \neq 0$.

Mumford's conjecture: *Reductive algebraic group G is Geometric reductivity*.

This was first proved for the case of $SL(2)$ (hence $GL(2)$) in characteristic 2 by Tadao Oda, and in all characteristics by Seshadri. W. Haboush proved the conjecture for a general reductive G in the 1974. Haboush's proof uses in an essential way the irreducibility of the Steinberg representation. A germ of this idea can perhaps be traced back to the appendix to Seshadri's paper by Raghunathan!

There is also a different approach to the problem due to Formanek and Procesi, à priori for the full linear group , but the general case can be deduced from this. Seshadri in the late 70's finally extended geometric reductivity over general excellent rings which is a basic tool for constructing moduli in mixed characteristics.

Seshadri (in the late 60's) wanted to prove the general Mumford conjecture using the geometric approach which was roughly: if the quotient set of "equivalence classes of semistable points for a linear action of G on a projective variety X" could be given the structure of a projective scheme then Mumford's conjecture holds for G, i.e. reductive = geometrically reductive.

Seshadri's paper on "Quotients modulo reductive groups" which has already been referred to before, has several beautiful ideas. He introduces the notion of "G-properness" which under some simple conditions shows that quotients, if they exist, are "proper and separated". One of the basic results in this paper is the following:

Let X be a projective variety on which there is given an action of a reductive algebraic group G with respect to an ample line bundle L on X. Let X^{ss} and X^{s} denote respectively *the semi-stable locus and the stable locus of the action of* G *on* (X, L)*. Suppose that* X *is normal,* $X^{ss} = X^s$*, and* G *acts freely on* X*. Then the geometric quotient* X^s/G *exists as a normal projective variety. Loosely put, this is Mumford's conjecture when "semistable = stable"*

Seshadri then gives a general technique to ensure the condition $X^{ss} = X^s$ can be made to hold. These have played a central role in several subsequent developments.

As we mentioned above, geometric reductivity of a reductive group G is equivalent to showing that the set Y of equivalence classes of semi-stable points for a linear action of G on a projective scheme X has a canonical structure of a projective scheme. The first difficulty is getting a natural scheme theoretic structure on Y. The second one, more difficult is to prove its projectivity.

When "stable = semi-stable" Seshadri showed that Y is a proper scheme and the proof reduces to checking the Nakai-Moishezon criterion for L on Y. This process led to Seshadri's ampleness criterion and Seshadri constants.

Around 2009, Seshadri and Pramath Sastry completed Seshadri's old argument. The key new ingredient (work of Sean Keel) was to be able to prove that under some conditions, line bundles which are “nef" and “big" are semi-ample. It was a recursive property for “nef" line bundles to become semi-ample, in a sense a “Nakai-Moisezon" for semi-ampleness.

Let L be semi-ample bundle on a scheme X (which we may assume is irreducible for simplicity) and as a consequence let $L^{\otimes n}$ be globally generated. Let $E(L)$ be the exceptional locus for the morphism $f : X \to | L^{\otimes n} |$. If L is also *big*, then the map f is a birational rational map (by definition where I assume $n \gg 0$).This ensures that $E(L) \subsetneq X$.On the other hand, Nakai-Moisezon says that if L is ample if and only if $E(L)$ is empty. *One of Keel's theorems (the main one I believe) is the following inductive process: let L be nef on X over a field of positive characteristics* *(it fails in characteristic zero!). Then L is semi-ample if and only if $L|_{E(L)}$ is semi-ample.*

Recent work of Jarod Alper on “Adequate moduli spaces", and the work of Alper, Halpern-Leitsner, Heinloth show how Seshadri’s work continues to inspire new work.

Seshadri was also an accomplished exponent of the Carnatic Music and till a few days before his passing, he continued to share his musical knowledge and insights with a young musical student from CMI. Seshadri was trained by his grandmother who herself was a well-trained singer. Seshadri showed the same traits in his musical discipline as in his mathematical ones. He meticulously did *riyaaz* and his repertoire in Muthuswamy Dikshitar's kritis and Shyama Sastry's kritis was noteworthy. I have had several occasions of listening to his music which can be described as a *royal gait without a trace of haste, as if he were in his true state.* While singing, a distinctly spiritual side of his used to come to the fore. By a spiritual side, I do not mean anything religious, but a musical one which bore the stamp of an immense *sadhana*, where every nuance was expressed with a spiritual feeling which was way beyond religious emotion.

I close my talk with lines from W.H. Auden:

"Like music when
Begotten notes, New notes beget.
Making the flowing of time a growing.
T'is what it could be....
When even sadness, Is a form of gladness.